

A Manifestly Noncovariant Theory of Gravitation

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We study a noncovariant theory of gravitation based on the Lagrangian density $(\sqrt{-g})^\omega R$, where ω is a constant. In particular, we study solutions that for $\omega = 1$ reduces to the de Sitter, Kasner and LFRW (with perfect fluid with as a source). We also consider spherically symmetric solutions.

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INTRODUCTION

Recently there has been a lot of attention to theories of gravitation that break the equivalence between the space and the time. Examples of these theories are: The TeVeS (Tensor Vector Scalar) [1] theory that is a covariant version of MOND (Modified Newtonian Dynamics) [2] that is thought as a modification of Newtonian gravitation in order to solve the problem of the galaxy rotation curves without the introduction of dark matter. In TeVeS a fundamental timelike vector field is introduced that gives a privileged time direction. The Einstein aether theory [3] where a dynamical unit timelike vector field is introduced to break local Lorentz symmetry. The main motivation for this theory is the suspicion that the vacuum in quantum gravity may determine a preferred rest frame at the microscopic level. The vector field defines a congruence of timelike curves filling all of space-time, like an omnipresent fluid, and so has been named “aether”. And the theory of gravity proposed by Horava [4], inspired by condensed matter models of dynamical critical systems. It has manifest three-dimensional spatial general covariance and time reparametrization invariance. It is described in the language of the Arnowitt-Deser-Misner (ADM) canonical Hamiltonian formulation of general relativity, but in which Einstein gravity is modified adding a new constant so that the full underlying four dimensional covariance is broken. But, it is restored at the infrared large distance limit, i.e., for a particular value of the above mentioned constant. The motivation for this theory is the construction of a renormalizable

theory in 3+1 dimensions. The relation between these last two theories is studied in [5].

The purpose of this letter is to study a new non-covariant modification of General Relativity (GR) that contains a new constant such that when the constant takes a particular value we recover the usual GR. Our starting point is the action for the gravity part of the theory,

$$S_G = \int (\sqrt{-g})^\omega (R + 2\Lambda) d^4x, \quad (1)$$

where Λ the cosmological constant and ω is a new constant. We use geometrical units $8\pi G = c = 1$ and metric signature $(+---)$. The previous action is invariant under a general change of coordinates only when $\omega = 1$. Therefore, for $\omega \neq 1$ the variation of this action yield noncovariant field equations. For the matter we choose the action,

$$S_M = \int (\sqrt{-g})^\sigma L_M d^4x, \quad (2)$$

where L_M is the usual Lagrangian that describes the matter content of the universe, and σ is a constant, we shall consider the two natural possibilities, that either σ takes the value 1 or ω . For the first value will have the usual coupling of matter used in GR and the second value is a natural choice compatible with the introduction of the parameter that breaks the covariance in (1). From the functional derivative, $\delta(S_G + 2S_M)/\delta g^{\mu\nu} = 0$, we find

$$R_{\mu\nu} - \frac{\omega}{2} R + \omega \Lambda g_{\mu\nu} + (\omega - 1) K_{\mu\nu} = -(\sqrt{-g})^{\sigma-\omega} T_{\mu\nu} - (1 - \sigma)(\sqrt{-g})^{\sigma-\omega} g_{\mu\nu} L_M, \quad (3)$$

where the object $K_{\mu\nu}$ is constructed with $g_{\mu\nu}$ and its first derivatives. We note that $(\sqrt{-g})^\omega R$ is linear in the second derivatives of the metric. So these second derivatives can be eliminated by the usual integration by parts. The explicit form of the object $K_{\mu\nu}$, that is not a tensor, is quite large and cumbersome and will be presented

elsewhere. The tensor $T_{\mu\nu}$ is the usual metric energy-momentum tensor $[\sqrt{-g}T_{\mu\nu} = 2\delta(\sqrt{-g}L_M)/\delta g^{\mu\nu}]$. From eq. (3) we see that when $\omega = 1$ we recover the usual GR (σ takes the values either one or ω). So we will pay special attention to the transition from $\omega \neq 1$, early universe, to $\omega = 1$, present era.

In this letter we shall consider three solutions with cosmological interest, a de Sitter like solution, a Kasner like solution, and a Lemaître-Friedman-Roberson-Walker (LFRW) like solution with a $p = \gamma\rho$ fluid as a source as well as some spherically symmetric solutions.

For the LFRW metric with flat spatial sections,

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \quad (4)$$

we find that the variation of S_G gives us,

$$2(3\omega - 2) \frac{d^2 a}{dt^2} a + (3\omega - 2)^2 \left(\frac{da}{dt} \right)^2 - \Lambda a^2 \omega = 0. \quad (5)$$

Looking for a solution of the form $a = e^{\alpha t}$, where α is a constant we get

$$\alpha = \sqrt{\frac{\Lambda}{3(3\omega - 2)}}. \quad (6)$$

The symmetry breaking parameter ω introduces a renormalization of the cosmological constant. Suppose that in the early universe ω is close to 2, in this case we can have an effective cosmological constant as large as wished. The for $\omega = 1$ we recover the usual expansion rate. This transition can give us a hint to solve the problem of the size of the cosmological constant computed in quantum field theory.

The Now we shall consider a Bianchi type I metric,

$$ds^2 = N^2(t)dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \quad (7)$$

In the study of the generic cosmological singularity of the solution of the Einstein equations this metric with $N = 1$ plays an essential role [6].

From $\delta S_G = 0$ we get

$$\begin{aligned} & \frac{da}{dt} \frac{db}{dt} c + \frac{da}{dt} \frac{dc}{dt} b + \frac{db}{dt} \frac{dc}{dt} a + \\ & \left(\frac{d^2 a}{dt^2} ab^2 c^2 + \left(\frac{da}{dt} \right)^2 b^2 c^2 + 3 \frac{da}{dt} \frac{db}{dt} abc^2 + 3 \frac{da}{dt} \frac{dc}{dt} ab^2 c + \frac{d^2 b}{dt^2} a^2 bc^2 + \left(\frac{db}{dt} \right)^2 a^2 c^2 + \right. \\ & \left. 3 \frac{db}{dt} \frac{dc}{dt} a^2 bc + \frac{d^2 c}{dt^2} a^2 b^2 c + \left(\frac{dc}{dt} \right)^2 a^2 b^2 \right) \frac{(\omega - 1)}{abc} = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} & Na^2 bc \left(\frac{dN}{dt} \frac{db}{dt} c + \frac{dN}{dt} \frac{dc}{dt} b - \frac{d^2 b}{dt^2} Nc - \frac{db}{dt} \frac{dc}{dt} N - \frac{d^2 c}{dt^2} Nb \right) + \\ & bc \left(- \frac{d^2 N}{dt^2} Na^2 bc + 2 \left(\frac{dN}{dt} \right)^2 a^2 bc + 2 \frac{dN}{dt} \frac{da}{dt} Nabc + \frac{dN}{dt} \frac{db}{dt} Na^2 c + \frac{dN}{dt} \frac{dc}{dt} Na^2 b - 2 \frac{d^2 a}{dt^2} N^2 abc + \right. \\ & \left(\frac{da}{dt} \right)^2 N^2 bc - 2 \frac{da}{dt} \frac{db}{dt} N^2 ac - 2 \frac{da}{dt} \frac{dc}{dt} N^2 ab - 2 \frac{d^2 b}{dt^2} N^2 a^2 c - 3 \frac{db}{dt} \frac{dc}{dt} N^2 a^2 - 2 \frac{d^2 c}{dt^2} N^2 a^2 b \Big) (\omega - 1) + \\ & \left(- \left(\frac{dN}{dt} \right)^2 a^2 b^2 c^2 - 2 \frac{dN}{dt} \frac{da}{dt} Nab^2 c^2 - 2 \frac{dN}{dt} \frac{db}{dt} Na^2 bc^2 - 2 \frac{dN}{dt} \frac{dc}{dt} Na^2 b^2 c - \left(\frac{da}{dt} \right)^2 N^2 b^2 c^2 - \right. \\ & \left. 2 \frac{da}{dt} \frac{db}{dt} N^2 abc^2 - 2 \frac{da}{dt} \frac{dc}{dt} N^2 ab^2 c - \left(\frac{db}{dt} \right)^2 N^2 a^2 c^2 - 2 \frac{db}{dt} \frac{dc}{dt} N^2 a^2 bc - \left(\frac{dc}{dt} \right)^2 N^2 a^2 b^2 \Big) (\omega - 1)^2 = 0, \end{aligned} \quad (9)$$

$$Do \quad a \rightarrow b \rightarrow c \rightarrow a \quad in \quad (9), \quad (10)$$

$$Do \quad a \rightarrow b \rightarrow c \rightarrow a \quad in \quad (10). \quad (11)$$

Now choosing the gauge $N = 1$ we have that these last

three equations reduce to

$$a^2 bc \left(\frac{d^2 b}{dt^2} c + \frac{db}{dt} \frac{dc}{dt} + \frac{d^2 c}{dt^2} b \right) +$$

$$bc \left(2 \frac{d^2 a}{dt^2} abc - \left(\frac{da}{dt} \right)^2 bc + 2 \frac{da}{dt} \frac{db}{dt} ac + 2 \frac{da}{dt} \frac{dc}{dt} ab + 2 \frac{d^2 b}{dt^2} a^2 c + 3 \frac{db}{dt} \frac{dc}{dt} a^2 + 2 \frac{d^2 c}{dt^2} a^2 b \right) (\omega - 1) +$$

$$\left(\left(\frac{da}{dt} \right)^2 b^2 c^2 + 2 \frac{da}{dt} \frac{db}{dt} abc^2 + 2 \frac{da}{dt} \frac{dc}{dt} ab^2 c + \left(\frac{db}{dt} \right)^2 a^2 c^2 + 2 \frac{db}{dt} \frac{dc}{dt} a^2 bc + \left(\frac{dc}{dt} \right)^2 a^2 b^2 \right) (\omega - 1)^2 = 0, \quad (12)$$

$$Do \quad a \rightarrow b \rightarrow c \rightarrow a \quad in \quad (12), \quad (13)$$

$$Do \quad a \rightarrow b \rightarrow c \rightarrow a \quad in \quad (13). \quad (14)$$

Now looking for a solution of the form $a = t^{p_1}, b = t^{p_2}, c = t^{p_3}$ we find

$$p_1 p_2 + p_2 p_3 + p_1 p_3 + (\omega - 1)(2p_1^2 + 2p_2^2 + 2p_3^2 - p_1 - p_2 - p_3 + 3p_1 p_2 + 3p_2 p_3 + 3p_3 p_1) = 0, \quad (15)$$

$$[\omega(p_1 + p_2 + p_3) - 1](p_i - p_j) = 0. \quad (i \neq j = 1, 2, 3) \quad (16)$$

Thus, the constants p_i satisfy,

$$\omega(p_1 + p_2 + p_3) = 1, \quad p_1^2 + p_2^2 + p_3^2 = (2\omega - 1)/\omega^2. \quad (17)$$

Solving the above equations in terms of p_1 , we find

$$2\omega p_2 = 1 - \omega p_1 + \sqrt{\Delta}, \quad (18)$$

$$2\omega p_3 = 1 - \omega p_1 - \sqrt{\Delta}, \quad (19)$$

$$\Delta \equiv -3[\omega P_1 - (1 - 2\sqrt{3\omega - 2/3})] \times$$

$$[\omega P_1 - (1 + 2\sqrt{3\omega - 2/3})]. \quad (20)$$

Thus to have a real solution we need $\omega \geq 2/3$ and

$$3\omega p_1 \geq 1 - 2\sqrt{3\omega - 2}, \quad 3\omega p_1 \leq 1 + 2\sqrt{3\omega - 2}. \quad (21)$$

We find for $\omega > 1$ a similar behavior than in GR, stretching and shrinking of the space in different directions governed by the signs of the exponents p_i . But, for $2/3 < \omega < 1$ we find that we can have that all the exponents positive, i.e., expansion in all directions, e.g., for $\omega \in [0.7, 0.8]$ and $p_1 \in [0.2, 0.3]$ we have $p_i > 0$ ($i = 1, 2, 3$). This suggest that near the generic singularity for $\omega \neq 1$ the spacetime will differ from the one of GR studied in [6].

Now we shall consider a LFRW universe filled with an irrotational perfect fluid with $p = \gamma\rho$ equation of state. The Lagrangian for this fluid can be expressed in terms of the velocity potential, Φ , as [7] $L_M = p = \frac{1}{2}(\partial^\alpha \Phi \partial_\alpha \Phi)^{(1+\gamma)/2\gamma}$. The four-velocity takes de form $u_\mu = \partial_\mu \Phi / \sqrt{\partial^\alpha \Phi \partial_\alpha \Phi}$. The variation of $S_G + 2S_M$ gives as

$$a^{3\sigma} \left(\frac{d\Phi}{dt} \right)^{\frac{1}{\gamma}} = K, \quad (22)$$

$$4(3\omega - 2)a^{3\omega+1} \frac{d^2 a}{dt^2} + 2(3\omega - 2)^2 a^{3\omega} \left(\frac{da}{dt} \right)^2$$

$$+ \sigma a^{3\sigma+2} \left(\frac{d\Phi}{dt} \right)^{\frac{1+\gamma}{\gamma}} = 0, \quad (23)$$

where K is an integration constant and Φ is a function of t only. Form the equations above and the ansatz $a = t^\alpha$ we find

$$2\alpha(3\alpha\omega - 2)(3\omega - 2)t^{3(\gamma\sigma+\omega)\alpha} + \sigma K^{1+\gamma} t^2 = 0. \quad (24)$$

Thus

$$\alpha = \frac{2}{3\gamma\sigma + 3\omega}, \quad K^{1+\gamma} = \frac{8(3\omega - 2)\gamma}{3(\gamma\sigma + \omega)^2}. \quad (25)$$

For the density and the Hubble “constant” ($H \equiv \frac{1}{a} \frac{da}{dt}$) we get, respectively,

$$\rho = \frac{4(3\omega - 2)}{3(\gamma\sigma + \omega)^2} t^{-\frac{2\sigma(1+\gamma)}{\gamma\sigma + \omega}}, \quad H = \frac{2}{3(\gamma\sigma + \omega)t}. \quad (26)$$

Therefore $\omega > 2/3$ to have a positive density. For this limit value of ω and $\sigma = \omega$ we have a expansion that is %50 greater that the predicted in GR. For $\omega > 1$ we have an expansion that is slower than in GR.

Now we shall study some spherically symmetric solutions of eq. (3), We shall consider the metric,

$$ds^2 = e^{A(r)} dt^2 - e^{B(r)} dr^2 - U^2(r)(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2). \quad (27)$$

For $A = B = 0$ the evolution equations reduce to

$$(4\omega - 3) \left(U \frac{d^2 U}{dr^2} + (\omega - 1) \left(\frac{dU}{dr} \right)^2 \right) + 1 - \omega = 0. \quad (28)$$

A solution to this equations is $U = r/\sqrt{4\omega - 3}$. We have found other exact solutions that we shall presented elsewhere. The Einstein tensor for this spacetime reduces

to $G_t^t = G_r^r = 4(\omega - 1)/r^2$. This spacetime in GR has already appeared and was related to a cloud of cosmic strings [8] and to monopoles [9].

Now we consider the case $U = r$. The variation of the action (1) yield,

$$\begin{aligned}
& 4 \left(-\frac{dB}{dr}r - e^B + 1 \right) + \left(4 \frac{d^2A}{dr^2}r^2 + \left(\frac{dA}{dr} \right)^2 r^2 - 2 \frac{dA}{dr} \frac{dB}{dr} r^2 + 8 \frac{dA}{dr}r + 2 \frac{d^2B}{dr^2}r^2 - \left(\frac{dB}{dr} \right)^2 r^2 - 4 \frac{dB}{dr}r - 4e^B \right. \\
& \left. + 12 \right) (\omega - 1) + \left(\left(\frac{dA}{dr} \right)^2 r^2 + 2 \frac{dA}{dr} \frac{dB}{dr} r^2 + 8 \frac{dA}{dr}r + \left(\frac{dB}{dr} \right)^2 r^2 + 8 \frac{dB}{dr}r + 16 \right) (\omega - 1)^2 = 0, \\
& 2 \left(\frac{dA}{dr}r - e^B + 1 \right) + \left(\frac{d^2A}{dr^2}r^2 + \left(\frac{dA}{dr} \right)^2 r^2 + 6 \frac{dA}{dr}r - 2e^B + 10 \right) (\omega - 1) = 0.
\end{aligned} \tag{29}$$

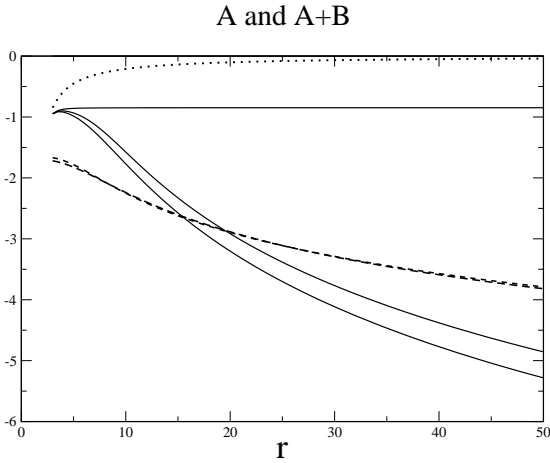


FIG. 1: The dotted line is the graph of $A(r)$ for $\omega = 1$, $A(r) + B(r) = 0$ in this case (Schwarzschild solution). The second line is A for $\omega = 1.5$, also $A + B = 0$ in this case. The next two curves represent A for $\omega = 1.6$ and 1.7 , respectively. The segmented lines are $(A + B)/5$ for the previous two values of ω , respectively.

First we note that doing $\omega = 1$ in eq. (29) we recover the two first order differential equations that have as solutions $e^A = e^{-B} = 1 - 2m/r$, i.e., the Schwarzschild solution. Since the second order derivatives disappear in this case we have that the Schwarzschild solution is a singular solution of (29). Using a fourth order Runge-Kutta algorithm we numerically solve the system (29) with the initial conditions $A(3) = -B(3) = \ln(3)$ and $dA(3)/dr = -dB(3)/dr = 2/3$. These conditions are obtained from A and B at $r = 3$ for the Schwarzschild solution with mass equal to one. In Fig. 1 we present the solution of the above mentioned system for different values of ω . The dotted line is the graph of $A(r)$ for $\omega = 1$, i.e., the Schwarzschild solution; $A(r) + B(r) = 0$ in this

case. The second line is A for $\omega = 1.5$, also $A + B = 0$ in this case. We note a very different behavior that in the precedent case, after $r = 10$ A is almost constant, we have a variation in the third decimal case only. The next two curves are A for $\omega = 1.6$ and 1.7 , they present a quite different behavior that the previous cases, first they are decreasing functions of r , and $A \neq -B$. The graph of the function $(A + B)/5$ for $\omega = 1.6$ and $\omega = 1.7$ are the two segmented lines, respectively. We have that after the first step of integration the value of B is quite different from its initial value (not shown in the figure) and the value of A is similar to its initial value. This system of equations will be further explored in another opportunity.

In summary, the theory based in the action (1) present some interesting features like the change of the rate of expansion for cosmological models. And it is simple enough to present analytical solutions for the most important cosmological models.

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- [1] J. D. Bekenstein, Phys. Rev. D 70, 083509 (2004)
- [2] M. Milgrom, Astrophysical Journal 270, 365 (1983)
- [3] C. Eling and T. Jacobson, Phys. Rev. D 69, 064005 (2004)
- [4] P. Hořava, Phys. Rev. D 79 084008 (2009)
- [5] T. Jacobson, Phys. Rev. D 81, 101502 (2010)
- [6] V.A. Belinskii, I.M. Khalatnikov, E.M. Lifshitz, Adv. Phys. 19, 525 (1970)
- [7] R. Tabensky, A. H. Taub, Commun. Math. Phys. 29 (1973),
- [8] P.S. Letelier Phys. Rev. D 20, 1294 (1979)
- [9] M. Barriola, A. Vilenkin, Phys. Rev. Lett. 63, 341 (1989)